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On the theory of localised snarling instabilities in false-twist yarn processes

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Abstract A theory for the twist-induced localised snarling instability observed in whirling and transported yarn in textile manufacturing processes such as false-twisting is developed. The buckling of the yarn can occur in two modes. At a critical level of the tension the straight yarn path bifurcates to a whirling ballooning mode. The localised snarling bifurcation can be triggered either from the straight line path prior to whirling or from the post-whirling configuration depending on the transport speed of the yarn through the system. The yarn is modelled as a pre-tensioned elastic rod. A perturbation analysis is carried out in which the small parameter measures bending relative to dynamical forces. The whirling bifurcation is captured with a regular perturbation analysis and the snarling bifurcation is captured with an internal bending layer in a singular perturbation analysis. This localised snarling is a subcritical bifurcation that occurs at a critical combination of yarn torque and tension. Critical conditions, as well as the position along the yarn where the snarling instability occurs, are obtained by matching the internal layer to the outer solutions. To accomplish this matching yarn axial elasticity is essential.

Keywords Elastic rods · False twist · Internal bending layer · Localised buckling · Singular perturbation · Textile-yarn snarling

1 Introduction

In this paper we derive a theory for the localised snarling instability that is some times observed in a ballooning or whirling yarn in textile-yarn manufacturing processes such as false-twisting [1, Sect. 2.6.2]. A photograph of a well-developed snarl is shown in Fig. 1. The mechanics of this well-developed snarl (or ply) has been the subject of previous studies (e.g., [2,3]). Here we consider the onset and initial stage of the snarling instability. We shall model the yarn as a long elastic rod of uniform circular cross-section, and uniform bending and torsional stiffness. The snarling instability can be treated as an internal bending layer that can be analysed using singular perturbation methods [4].

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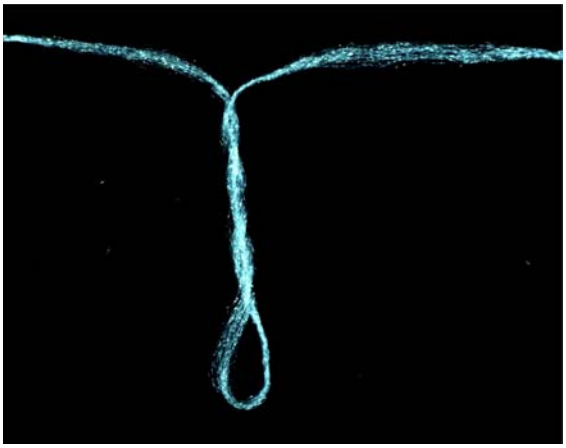

Fig. 1 Photograph of a well-developed snarl in false-twist textured yarn. (Image provided by David Phillips, Commonwealth Scientific and Industrial Research Organisation (CSIRO), Textile & Fibre Technology, Geelong (Vic), Australia.)

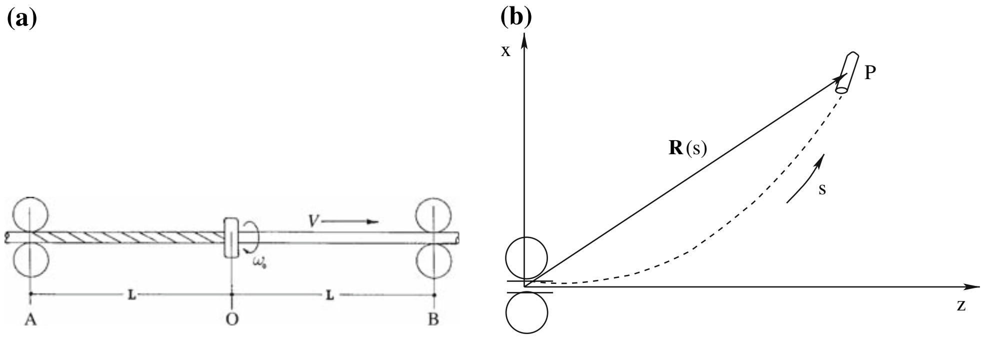

Fig. 2 The false-twist process. (**a**) Schematic diagram of the generic process. (**b**) Notation

The theory given here uses an elasto-dynamic rod model for twist insertion in textile yarn. The false-twist system to be analysed is shown schematically in Fig. 2a. Twist is inserted between the roller nip A and the twisting spindle O, and is removed between O and the exit rollers B, so that the yarn exiting from these rollers is without twist. At A a *spinning triangle* [5] is formed in which, over a very short distance, the individual strands or fibres that make up the yarn are wrapped into the 'rod' structure and accelerated so that, over the rest of its length, the yarn following its steady, prebuckling, straight-line path in the uptwist region, is rotating with the uniform angular speed ω_0 of the spindle at O. Thus, between A and O the yarn is subject to a high constant torque, and between O and B the torque is negligible and will be assumed zero in this analysis [6]. As in [7] we will ignore the length of the spinning triangle and take the origin of our coordinate system at the inlet roller nip A. The analysis presented here is confined to the uptwist region between A and O where torque is high and snarling instabilities are likely to occur. It will be assumed that no slippage occurs at the entry roller or the twisting-spindle. In normal operation, before buckling occurs, the yarn follows a straight path in this region, and is subject to a tension T_0. The pre-stretch caused by this tension will not be considered in this analysis as buckling is assumed to take place about this steady-state prebuckled configuration. Variations in tension $\tilde{T}$ due to the buckling are assumed to be small compared with T_0. Additional strains due to $\tilde{T}$ are assumed to be a linear function of $\tilde{T}$. The analysis in the first part of this paper is similar to that given in the paper by Zhu et al. [8], and we use similar dimensionless quantities to theirs. These authors model the yarn as a string, which allows them to study yarn ballooning but not the subsequent snarling for which the bending and torsional stiffness of the yarn is essential.

The independent variables in the theory are the distance s, measured along the axis of the pre-stretched yarn, to a material element of the yarn (P in Fig. 2b) from the inlet roller nip A; and the time t. The distance variable s is thus a Lagrangian variable so that, if we assume that the mass linear density of the pre-stretched pre-buckled yarn is uniform (i.e., $m(s) = m$ is constant), the use of this independent variable means that the mass of P does not change during the motion, although the physical length of the element ($\delta\ell$ say) will be

$$\delta\ell = \left(1 + \frac{\tilde{T}}{AE}\right)\delta s. \tag{1.1}$$

The total tension is given by

$$T = T_0 + \tilde{T}. \tag{1.2}$$

A is the cross-sectional area of the yarn, assumed constant, and E is the tangent modulus of the yarn at the point of its stress/strain curve consistent with the pre-stretch caused by the pre-tension T_0. The bending and torsional stiffnesses B and K will be assumed constant and independent of tension $\tilde{T}$ and torque Q. We will assume that the localised buckling occurs in the uptwist region and that any variation in the twist due to buckling is small so that the strains due to twist variation will be ignored.

Other assumptions are as follows. The linear speed $\tilde{V}$ of the yarn through the system is slow compared with the speed of extensional and torsional waves in the yarn. As shown by Miao and Chen [9], who seem to be the first authors to propose an elastic-rod model for twist insertion in textile yarn (albeit for straight yarn paths), if $\tilde{V}$ exceeds the speed of torsional waves no twist can be inserted between the feed rollers and the spindle. Also, it is assumed that the yarn does not contact any machine surfaces between the feed rollers and the spindle. The action of air-drag on the yarn is also assumed to be negligible.

The feed speed $\tilde{V}$ is assumed to be constant so that in time interval δt the pre-stretched distance along the yarn axis from the roller nip to P increases by $\delta s = \tilde{V}\delta t$. The material derivative operator is thus given by

$$\mathrm{D}(\,) = \frac{\partial(\,)}{\partial t} + \tilde{V}\frac{\partial(\,)}{\partial s}. \tag{1.3}$$

The rod theory is based on the usual assumption that plane sections remain plane and perpendicular to the rod axis and that the bending-moment/curvature and torque/torsion constitutive relations are linear (Eq. 2.2 below).

The organisation of the paper is as follows. In Sect. 2 the equations for a transported rotating elastic rod are set up and an appropriate nondimensionalisation is introduced that identifies the ratio of the bending force to the dynamical force that drives the system as a small parameter. In Sect. 3 a regular perturbation expansion picks up the ballooning of the yarn. This solution is used in Sect. 4 for the outer regions of the balloon while an internal layer is constructed to describe the twist-induced snarling instability. This internal layer is governed by the equations for a stationary free inextensible rod. The relevant solution here is the localised buckling solution studied in [10,11]. The localised buckling involves significant stretching of the yarn, which in this model is accommodated by the yarn elasticity. Thus the buckling of the yarn is seen to be a two-stage process. At a critical level of T_0 the straight yarn path bifurcates to a whirling ballooning mode and the localised snarling bifurcates from this whirling mode at a critical combination of Q and T_0. Snarling effectively proceeds under dead-loading conditions and involves a jump into self-contact. Under certain conditions snarling may also occur directly from the straight yarn path. Critical conditions, as well as the position along the yarn where the snarling instability occurs, are obtained by matching inner and outer solutions. To accomplish this matching the yarn axial elasticity is essential.

2 The mathematical formulation

As discussed above, in this paper the yarn is modelled as an elastic rod of uniform circular cross-section of radius r and mass per unit length m. The implications of these assumptions for modelling textile-yarn dynamics have been discussed in detail in [7]. The torque/torsion, and bending-moment/curvature constitutive equations are assumed to be linear. The bending and torsional stiffnesses are given by

$$K = \frac{1}{2}GAr^2, \quad \text{and} \quad B = \frac{1}{4}EAr^2,$$

where E is the tangent modulus, G is the shear modulus, and $A = \pi r^2$ is the cross-sectional area of the rod [12, Chapts. 4 and 5]. If ν is Poisson's ratio, then $G = E/[2(1+\nu)]$, and with $\nu = 0.5$ (the incompressible limit) this leads to

$$K = \frac{2}{3}B.$$

The point to be made here is that these stiffnesses have the same order of magnitude. These formulae are also used below to estimate this order of magnitude. Data given in [13], and more recently in [14], show that the ratio K/B given above is in approximate agreement with their data for textile yarns.

If the position vector of a material element P (Fig. 2b) of the thread line at time t with respect to the feed roller nip at A is $\mathbf{R}(s,t)$, then the extensibility and unshearability conditions are

$$\mathbf{R}' \cdot \mathbf{R}' = \left(1 + \frac{\tilde{T}}{AE}\right)^2, \quad \text{and} \quad \mathbf{t} \cdot \mathbf{V} = 0, \tag{2.1}$$

and the bending moment/curvature and torque/torsion constitutive equations are

$$\mathbf{M} = B(\mathbf{t} \times \mathbf{t}'), \quad Q = K\tau. \tag{2.2}$$

In the above equations $\mathbf{V}$ and $\mathbf{M}$ are, respectively, the shear-force and bending-moment vectors acting on a rod cross-section, and $(\ )' = \partial(\ )/\partial s$. The tangent vector, principal normal and binormal are given by

$$\mathbf{t} = \frac{\mathbf{R}'}{(1 + \tilde{T}/AE)}, \quad \mathbf{n} = \mathbf{t}'/|\mathbf{t}'| = \mathbf{t}'/\hat{\kappa}, \quad \text{and} \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}, \tag{2.3}$$

where $\hat{\kappa}$ is the curvature of the yarn path. In discussing elastic-rod models of thread-line dynamics, it is convenient to introduce the torsion τ defined in [15, Chapt. 18] as

$$\tau = \phi' + \mathbf{b} \cdot \mathbf{n}' = \phi' + (\mathbf{b} \cdot \mathbf{t}'')/\hat{\kappa}. \tag{2.4}$$

The contribution $(\mathbf{b} \cdot \mathbf{n}')$ to τ is the *tortuosity* of the rod axis, and ϕ is the angle between the plane of curvature (defined by the vectors $\mathbf{t}$ and $\mathbf{n}$) and a radial line, perpendicular to the strand axis, joined to any straight line parallel to the axis, marked on the surface of the initially straight untwisted rod; ϕ' can be interpreted as the initial torsion in the straight strand before its axis is deformed into a curved path.

2.1 The equations of motion

We write the equations of motion relative to an orthonormal reference frame $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, that rotates with constant angular velocity $\omega_0\mathbf{k}$ where $\mathbf{k}$ is directed from A along the prebuckled yarn path towards the spindle; ω_0 is the angular speed the twisting spindle gives to the yarn, assumed constant.

With respect to this axis system the equation for the rate of change of linear momentum of the yarn element P is

$$m\left[D^2\mathbf{R} + 2\omega_0\mathbf{k} \times D\mathbf{R} + \omega_0^2\mathbf{k} \times (\mathbf{k} \times \mathbf{R})\right] = (T\mathbf{t} + \mathbf{V})'. \tag{2.5}$$

If the angular velocity of the element P is $\mathbf{\Omega}(s,t)$, an expression for this variable in terms of the unit tangent vector $\mathbf{t}$ is derived as follows:

$$D\mathbf{t} + \omega_0\mathbf{k} \times \mathbf{t} = \mathbf{\Omega} \times \mathbf{t},$$

so that

$$\mathbf{t} \times (D\mathbf{t} + \omega_0\mathbf{k} \times \mathbf{t}) = \mathbf{t} \times (\mathbf{\Omega} \times \mathbf{t}) = \mathbf{\Omega} - (\mathbf{\Omega} \cdot \mathbf{t})\mathbf{t},$$

on expansion of the vector triple product on the right. On rearrangement this gives

$$\mathbf{\Omega} = \omega_t\mathbf{t} + \mathbf{t} \times (D\mathbf{t} + \omega_0\mathbf{k} \times \mathbf{t}), \tag{2.6}$$

where $\omega_t = \mathbf{\Omega} \cdot \mathbf{t}$ is the component of the yarn angular velocity about the yarn axis.

The principal moments of inertia of the element are

$$I_{\text{axial}} = \frac{1}{2} m r^2 \delta s \quad \text{and} \quad I_{\text{dia}} = \frac{1}{4} m r^2 \delta s = \frac{1}{2} I_{\text{axial}}.$$

Thus, the total angular momentum of the element about its centre of mass is

$$\mathbf{H}\delta s = \frac{1}{2} m r^2 \delta s \left\{ \omega_t \mathbf{t} + \frac{1}{2} \left[\mathbf{t} \times (D\mathbf{t} + \omega_0 \mathbf{k} \times \mathbf{t}) \right] \right\}. \tag{2.7}$$

Finally the rate of change of this angular momentum relative to the inertial frame is equal to the resultant moment of all the forces and moments about the centre of mass acting on the element:

$$\begin{aligned} D\mathbf{H} + \omega_0 \mathbf{k} \times \mathbf{H} = \frac{1}{2} m r^2 \Big\{ &(D\omega_t)\mathbf{t} + \omega_t D\mathbf{t} + \omega_0 \omega_t (\mathbf{k} \times \mathbf{t}) \\ &+ \frac{1}{2} \left[\mathbf{t} \times D^2 \mathbf{t} - 2\omega_0 (\mathbf{k} \cdot \mathbf{t}) D\mathbf{t} - \omega_0^2 (\mathbf{k} \cdot \mathbf{t})(\mathbf{k} \times \mathbf{t}) \right] \Big\} = (Q\mathbf{t})' + \mathbf{M}' + \mathbf{R}' \times \mathbf{V}. \end{aligned} \tag{2.8}$$

Note that the equations of motion (2.5) and (2.8) do not depend on the particularity of the bending moment/curvature or torque/torsion constitutive relations.

The **t**-component of (2.8) is

$$\frac{1}{2} m r^2 D\omega_t = Q', \tag{2.9}$$

an equation which depends on the yarn path only through the torque/torsion constitutive equation. When this axial component of the angular momentum equation is subtracted from (2.8) the transverse component of (2.8) is obtained:

$$\begin{aligned} &\frac{1}{2} m r^2 \left\{ \omega_t D\mathbf{t} + \omega_0 \omega_t (\mathbf{k} \times \mathbf{t}) + \frac{1}{2} \left[\mathbf{t} \times D^2 \mathbf{t} - 2\omega_0 (\mathbf{k} \cdot \mathbf{t}) D\mathbf{t} - \omega_0^2 (\mathbf{k} \cdot \mathbf{t})(\mathbf{k} \times \mathbf{t}) \right] \right\} \\ &\quad = Q\mathbf{t}' + \mathbf{M}' + \mathbf{R}' \times \mathbf{V}. \end{aligned} \tag{2.10}$$

2.2 Boundary conditions

In the light of the above discussion the boundary conditions at the inlet rollers and the spindle are

$$\mathbf{R}(0, t) = \mathbf{0}, \quad \mathbf{R}(L, t,) = L\mathbf{k}, \quad \mathbf{M}(0, t) = \mathbf{M}(L, t) = \mathbf{0}, \tag{2.11}$$

L being the length of the uptwist region.

2.3 Dimensionless equations

We shall use L as the length scale, $1/\omega_0$ as the time scale, and forces will be made dimensionless with respect to $m\omega_0^2 L^2$. Thus dimensionless (barred) variables are defined as follows:

$$\left.\begin{aligned} &(\bar{\mathbf{R}}, \bar{s}) = \frac{(\mathbf{R}, s)}{L}, \quad \bar{t} = \omega_0 t, \quad \bar{\omega} = \frac{\omega_t}{\omega_0}, \quad \bar{T}_W = L T_W, \quad \epsilon^2 = \frac{B}{m\omega_0^2 L^4}, \quad \kappa = \frac{K}{B}, \\ &\gamma = \frac{AE}{m\omega_0^2 L^2}, \quad \frac{r}{L} = \epsilon \frac{2}{\sqrt{\gamma}}, \quad \epsilon \mathcal{V} = \frac{\tilde{V}}{\omega_0 L}, \quad \bar{D} = \frac{D}{\omega_0} = \left(\frac{\partial}{\partial \bar{t}} + \epsilon \mathcal{V} \frac{\partial}{\partial \bar{s}} \right), \\ &(\bar{T}, \bar{\mathbf{V}}) = \frac{(T, \mathbf{V})}{m\omega_0^2 L^2}, \quad (\bar{Q}, \bar{\mathbf{M}}) = \frac{(Q, \mathbf{M})}{m\omega_0^2 L^3}, \quad \bar{\tau} = L\tau, \end{aligned}\right\} \tag{2.12}$$

where the small parameter $\epsilon \approx 10^{-3}$–10^{-5}, and κ, $\mathcal{V}$ and γ are $O(1)$ quantities. Typical parameter values are listed in Table 1.

Table 1 Typical dimensional and dimensionless parameters for the false-twist process [1]

L	4 m	ϵ	0.000024
r	0.0005 m	$\mathcal{V}$	10.43
m	167×10^{-7} kg/m	γ	0.15
ω_0	10,000 rad/s	κ	2/3
$\tilde{V}$	10.0 m/s		
E	5×10^9 N/m^2		

As all variables will be dimensionless from now on, unless specifically stated otherwise, the barred notation will be dropped.

The equation for the rate of change of linear momentum (2.5) becomes

$$D^2\mathbf{R} + 2(\mathbf{k} \times D\mathbf{R}) + \mathbf{k} \times (\mathbf{k} \times \mathbf{R}) = (T\mathbf{t} + \mathbf{V})', \tag{2.13}$$

where $T = T_0 + \tilde{T}$. Those for the rate of change of angular momentum (2.8) and (2.10) become

$$\epsilon^2 \frac{2}{\gamma} D\omega = Q', \tag{2.14}$$

and

$$\epsilon^2 \frac{2}{\gamma} \left\{ \left[\omega - \frac{1}{2}(\mathbf{k} \cdot \mathbf{t}) \right] (\mathbf{k} \times \mathbf{t}) + [\omega - (\mathbf{k} \cdot \mathbf{t})] D\mathbf{t} + \frac{1}{2}(\mathbf{t} \times D^2\mathbf{t}) \right\} = Q\mathbf{t}' + \mathbf{M}' + \mathbf{R}' \times \mathbf{V}, \tag{2.15}$$

subject to the constraints

$$\mathbf{t} = \frac{\mathbf{R}'}{(1 + \tilde{T}/\gamma)}, \quad \mathbf{R}' \cdot \mathbf{R}' = (1 + \tilde{T}/\gamma)^2 \quad \text{and} \quad \mathbf{V} \cdot \mathbf{t} = 0. \tag{2.16}$$

The constitutive equations (2.2) become

$$\mathbf{M} = \epsilon^2(\mathbf{t} \times \mathbf{t}'), \quad Q = \epsilon^2 \kappa \tau. \tag{2.17}$$

Equations (2.13) through (2.17) together with an appropriate set of boundary and initial conditions constitute a well-posed system of equations.

2.3.1 Twist

In dimensional terms textile-yarn twist T_W is measured in turns per metre of straight yarn. Thus $2\pi T_W = \phi'$. False twist systems insert a precisely determined amount of twist into the yarn which is calculated as follows. Consider the rotation of a yarn cross-section in time δt, which is $(\omega_0/2\pi)\delta t$, as an additional length of yarn $\tilde{V}\delta t$ is fed through the inlet roller nip. The twist inserted is

$$T_W = \frac{\omega_0 \delta t}{2\pi \tilde{V} \delta t} = \frac{\omega_0}{2\pi \tilde{V}}. \tag{2.18}$$

In topological terms we are inserting *Link* into the yarn and the amount of Link inserted into the straight yarn in the uptwist region is just

$$Lk = T_W \times L, \tag{2.19}$$

and since the system runs at constant speeds $\tilde{V}$ and ω_0 this quantity is unchanged, even when whirling and snarling occur.

In dimensionless terms Eq. 2.18 becomes

$$T_W = 1/(2\pi\epsilon\mathcal{V}), \tag{2.20}$$

and the constitutive equation $(2.17)_2$ can now be written as

$$Q = \epsilon^2 \kappa \left(2\pi T_W + \mathbf{b} \cdot \mathbf{n}'\right) \epsilon \frac{\kappa}{\mathcal{V}} + \epsilon^2 \kappa \mathbf{b} \cdot \mathbf{n}'. \tag{2.21}$$

Finally, introducing the constitutive relations (2.17)$_1$ and (2.21) into the transverse component of the equation for the rate of change of angular momentum (2.14), we obtain

$$\epsilon^2 \frac{2}{\gamma}\left\{[\omega - \frac{1}{2}(\mathbf{k}\cdot\mathbf{t})](\mathbf{k}\times\mathbf{t}) + [\omega - (\mathbf{k}\cdot\mathbf{t})]D\mathbf{t} + \frac{1}{2}(\mathbf{t}\times D^2\mathbf{t})\right\}$$
$$= \epsilon\frac{\kappa}{\mathcal{V}}\mathbf{t}' + \epsilon^2\kappa(\mathbf{b}\cdot\mathbf{n}')\mathbf{t}' + \epsilon^2(\mathbf{t}\times\mathbf{t}'') + \mathbf{R}'\times\mathbf{V}. \tag{2.22}$$

2.3.2 Boundary conditions

The dimensionless form of the boundary conditions is

$$\mathbf{R}(0,t) = \mathbf{0}, \quad \mathbf{R}(1,t) = \mathbf{k}, \quad \mathbf{M}(0,t) = \mathbf{M}(1,t) = \mathbf{0}. \tag{2.23}$$

3 The regular perturbation analysis—ballooning

We expand the variables in the following perturbation series:

$$\mathbf{R} = \mathbf{k}s + \epsilon\mathbf{R}_1 + \epsilon^2\mathbf{R}_2 + \cdots, \quad \mathbf{V} = \mathbf{V}_0 + \epsilon\mathbf{V}_1 + \epsilon^2\mathbf{V}_2 + \cdots, \quad \tilde{T} = \epsilon T_1 + \epsilon^2 T_2 + \cdots, \tag{3.1}$$

and let $\mathbf{R}_1 = u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}$, and so on. Now introduce these expansions into (2.16) to obtain

$$T_1 = \gamma w_1', \quad T_2 = \gamma\left[w_2' + \frac{1}{2}({u_1'}^2 + {v_1'}^2)\right], \tag{3.2,3.3}$$

and the expansion of the tangent vector

$$\mathbf{t} = \mathbf{k} + \epsilon(u_1'\mathbf{i} + v_1'\mathbf{j}) + \epsilon^2\left\{(u_2' - w_1'u_1')\mathbf{i} + (v_2' - w_1'v_1')\mathbf{j} - \frac{1}{2}({u_1'}^2 + {v_1'}^2)\mathbf{k}\right\} + \cdots. \tag{3.4}$$

It can also be shown that the tortuosity $(\mathbf{b}\cdot\mathbf{n}') \sim O(\epsilon^3)$.

The expansion for the shear force $\mathbf{V}$ is found as follows. Form the vector product of $\mathbf{t}$ with the angular momentum equation (2.15)$_2$ and introduce the above expansions into this result and the constraint equation (2.16)$_3$ to determine that

$$\mathbf{V}_0 = \mathbf{V}_1 = \mathbf{0}, \tag{3.5}$$

and the leading-order non-zero contribution from the transverse angular momentum equation is

$$\mathbf{V}_2 = \frac{\kappa}{\mathcal{V}}(-v_1''\mathbf{i} + u_1''\mathbf{j}). \tag{3.6}$$

With these results we can now write down the expansion of the equation of linear momentum (2.13) up to $O(\epsilon^2)$ as follows.

3.1 The $O(\epsilon)$ equations

The Cartesian components of the $O(\epsilon)$ linear momentum equation are

$$\frac{\partial^2 u_1}{\partial t^2} - 2\frac{\partial v_1}{\partial t} - u_1 - T_0 u_1'' = 0, \tag{3.7}$$
$$\frac{\partial^2 v_1}{\partial t^2} + 2\frac{\partial u_1}{\partial t} - v_1 - T_0 v_1'' = 0, \tag{3.8}$$
$$\frac{\partial^2 w_1}{\partial t^2} = \gamma w_1'', \tag{3.9}$$

subject to the boundary conditions

$$u_1(0,t) = v_1(0,t) = w_1(0,t) = 0, \quad u_1(1,t) = v_1(1,t) = w_1(1,t) = 0. \tag{3.10}$$

We first look for solutions to Eqs. 3.7 and 3.8 that represent the lowest mode of ballooning in the uptwist region:

$$u_1 = \left(U \sin\lambda t + \bar{U}\cos\lambda t\right)\sin\pi s, \quad v_1 = \left(V\cos\lambda t + \bar{V}\sin\lambda t\right)\sin\pi s. \tag{3.11}$$

There are two cases to be considered.

Case 1

$$\begin{aligned} &V = U, \quad \bar{V} = -\bar{U}, \quad T_0\pi^2 = (\lambda-1)^2, \\ &u_1 = \left(U\sin\lambda t + \bar{U}\cos\lambda t\right)\sin\pi s, \quad v_1 = \left(U\cos\lambda t - \bar{U}\sin\lambda t\right)\sin\pi s. \end{aligned} \tag{3.12}$$

Case 2

$$\begin{aligned} &V = -U\bar{V} = \bar{U}, \quad T_0\pi^2 = (\lambda+1)^2, \\ &u_1 = \left(U\sin\lambda t + \bar{U}\cos\lambda t\right)\sin\pi s, \quad v_1 = -\left(U\cos\lambda t - \bar{U}\sin\lambda t\right)\sin\pi s. \end{aligned} \tag{3.13}$$

These two solutions differ only in the rotational direction of the whirling. They correspond to stationary whirl in the fixed frame, with angular velocity $\pi\sqrt{T_0}$. The special case $T_0 = 1/\pi^2$ and $\lambda = 0$ represents a yarn path that is stationary relative to the rotating reference frame.

We note that the solution of (3.9) is independent of these solutions, and in both cases we may write the solution as

$$w_1 = \left(W\sin\pi\sqrt{\gamma}t + \bar{W}\cos\pi\sqrt{\gamma}t\right)\sin\pi s, \tag{3.14}$$

where again, for simplicity of presentation, we shall only consider the lowest longitudinal mode of vibration. To further fix the values of T_0 and λ at which this ballooning mode bifurcates from the straight yarn path, we must investigate the conditions for the existence of a solution to the $O(\epsilon^2)$ equations.

3.2 The $O(\epsilon^2)$ equations

The Cartesian components of the $O(\epsilon^2)$ linear momentum equation are

$$\begin{aligned} &\frac{\partial^2 u_2}{\partial t^2} - 2\frac{\partial v_2}{\partial t} - u_2 - T_0 u_2'' = -2\mathcal{V}\frac{\partial}{\partial s}\left(\frac{\partial u_1}{\partial t} - v_1\right) + (\gamma - T_0)(w_1' u_1')' - \frac{\kappa}{\mathcal{V}}v_1''', \\ &\frac{\partial^2 v_2}{\partial t^2} + 2\frac{\partial u_2}{\partial t} - v_2 - T_0 v_2'' = -2\mathcal{V}\frac{\partial}{\partial s}\left(\frac{\partial v_1}{\partial t} + u_1\right) + (\gamma - T_0)(w_1' v_1')' + \frac{\kappa}{\mathcal{V}}u_1''', \\ &\frac{\partial^2 w_2}{\partial t^2} - \gamma w_2'' = -2\mathcal{V}\frac{\partial^2 w_1}{\partial s\partial t} + \frac{1}{2}(\gamma - T_0)(u_1'^2 + v_1'^2)'. \end{aligned} \tag{3.15}$$

When the results of Sect. 3.1 are inserted in the right side of $(3.15)_3$ we obtain

$$\begin{aligned} \frac{\partial^2 w_2}{\partial t^2} - \gamma w_2'' = &-2\mathcal{V}\left[\pi^2\sqrt{\gamma}(W\cos\pi\sqrt{\gamma}t - \bar{W}\sin\pi\sqrt{\gamma}t)\cos\pi s\right] \\ &-\frac{1}{2}(\gamma - T_0)(U^2 + \bar{U}^2)\pi^3\sin 2\pi s. \end{aligned}$$

In order to obtain a solution of this equation, the terms on the right-hand side that resonate with the solution of the corresponding homogeneous equation must be eliminated. In this case we require that $W = \bar{W} = 0$ so that the $O(\epsilon)$ solution is $w_1 \equiv 0$ and hence $T_1 \equiv 0$ by (3.2). Thus there are no longitudinal vibrations of the yarn at this order. When the resonant terms on the right of the other two equations of $(3.15)_{1,2}$ are eliminated we find that for both *cases 1* and *2* above the critical value of the tension is

$$T_0 = \left(\frac{\kappa\pi}{2\mathcal{V}^2}\right)^2, \tag{3.16}$$

and

$$\lambda = \pm\left(1 + \frac{\kappa\pi^2}{2\mathcal{V}^2}\right), \tag{3.17}$$

where in this latter expression the upper sign corresponds to *case 1* and the lower sign to *case 2*.

At the critical tension given by (3.16) the first-mode ballooning solution bifurcates from the straight state. Higher-order modes similarly bifurcate at loads obtained by replacing π in (3.16) by $n\pi$.

Finally we note that the right sides of $(3.15)_{1,2}$ are now reduced to zero so that the only solutions of these equations will be proportional to the solutions of the $O(\epsilon)$ equations. Thus, without loss of generality, we can set $u_2 = v_2 \equiv 0$, and the nonzero parts of the $O(\epsilon^2)$ solutions are

$$w_2 = -\frac{(\gamma - T_0)\pi}{8\gamma}(U^2 + \bar{U}^2)\sin 2\pi s, \tag{3.18}$$

$$T_2 = (U^2 + \bar{U}^2)\frac{\pi^2}{4}(T_0 \cos 2\pi s + \gamma). \tag{3.19}$$

This positive T_2 describes the increase in yarn tension due to ballooning.

We now consider the further localised buckling bifurcation from this ballooning solution.

4 The localised snarling instability

In this section we obtain a solution that represents a localised buckling instability of the yarn. Such instabilities, in which the yarn develops a local buckle which rapidly increases in amplitude until the yarn makes self contact and snarls, are often observed in practice (see Fig. 1). This snarling instability imposes a serious limitation on the speed of yarn-twisting processes. We consider the possibility of a localised instability that occurs on the whirling (or ballooning) yarn configuration analysed in the last section. Here we shall give the detailed analysis for the *case 1* pre-buckled solution above. The *case 2* solution is obtained by reversing the sign on v_1. Thus we will assume that the localised buckle is centred at a point $s = s_b$ along the pre-buckled yarn path that has the position vector

$$\begin{aligned}\mathbf{R}(s_b, t) = s_b\mathbf{k} + \epsilon\left[\left(U\sin\lambda t + \bar{U}\cos\lambda t\right)\mathbf{i} + \left(U\cos\lambda t - \bar{U}\sin\lambda t\right)\mathbf{j}\right]\sin\pi s_b \\ -\epsilon^2\frac{(\gamma - T_0)\pi}{8\gamma}(U^2 + \bar{U}^2)\sin 2\pi s_b\mathbf{k} + O(\epsilon^3).\end{aligned} \tag{4.1}$$

Note that there is no axial yarn vibration in the $O(\epsilon)$ ballooning solution.

We analyse this instability as an internal bending layer using the method of singular perturbation expansions [4].

The perturbation equations above represent the outer expansion except that now we must carry out the analysis of these equations in three separate regions as follows:

- Region 1: where $0 \leq s < s_b$, which is the part of the outer solution between the guide-eye and the internal layer in which bending is very small;
- Region 2: the internal buckling layer where the curvature of the yarn axis is large and bending becomes significant. The torque, which is determined by the outer solution, remains constant through this region;
- Region 3: where $s_b < s \leq 1$, which is the part of the outer solution between the internal layer and the twisting spindle.

4.1 The outer solution

We now seek the solutions of the $O(\epsilon)$ equations (3.7)–(3.9) in regions 1 and 3. Again there are two cases to be considered, and we shall confine our exposition to the solutions corresponding to *case 1* above.

Region 1 $(0 \le s < s_b)$:

$$u_1 = \left(U_1 \sin\lambda_1 t + \bar{U}_1 \cos\lambda_1 t\right)\sin\alpha s, \quad v_1 = \left(U_1 \cos\lambda_1 t - \bar{U}_1 \sin\lambda_1 t\right)\sin\alpha s,$$

$$w_1 = as, \quad T_1 = \gamma a, \tag{4.2}$$

where

$$T_0\alpha^2 = (\lambda_1 - 1)^2. \tag{4.3}$$

Region 3 $(s_b < s \le 1)$:

$$u_1 = \left(U_3 \sin\lambda_3 t + \bar{U}_3 \cos\lambda_3 t\right)\sin\beta(1-s), \quad v_1 = \left(U_3 \cos\lambda_3 t - \bar{U}_3 \sin\lambda_3 t\right)\sin\beta(1-s),$$

$$w_1 = b(1-s), \quad T_1 = -\gamma b, \tag{4.4}$$

where

$$T_0\beta^2 = (\lambda_3 - 1)^2. \tag{4.5}$$

These solutions also satisfy the boundary conditions at $s = 0, 1$. The constants a and b in the expressions for w_1 and T_1 above represent a static component of the axial yarn strain. This component of the strain is essential in matching the additional stretching caused by the localised buckling deformation. The constants $\lambda_1, \lambda_3, a, b, \alpha, \beta$ are to be determined in part from the matching procedure described below and in Appendix A.

4.2 The inner solution

The internal bending layer occurs at position $\mathbf{R}(s_b, t)$ given by Eq. 4.1 above. Thus we define a stretched coordinate and inner expansions as follows:

$$\eta = \frac{(s - s_b)}{\epsilon}, \quad \mathbf{R} = \mathbf{R}(s_b, t) + \epsilon\mathbf{r}_0(\eta, t) + \epsilon^2\mathbf{r}_1(\eta, t) + \cdots, \tag{4.6}$$

$$\mathbf{V} = \hat{\mathbf{V}}_0(\eta, t) + \epsilon\hat{\mathbf{V}}_1(\eta, t) + \cdots, \quad T = \hat{T}_0 + \epsilon\hat{T}_1(\eta, t) + \epsilon^2\hat{T}_2(\eta, t) + \cdots, \tag{4.7}$$

where a ‘hat’ has been used to distinguish the inner variables other than $\mathbf{r}(\eta)$. With this substitution, the operator D becomes

$$\hat{\mathrm{D}} = \frac{\partial(\,)}{\partial t} + \mathcal{V}\frac{\partial(\,)}{\partial\eta}. \tag{4.8}$$

First we determine the torque in the inner solution. In terms of the inner variables the axial component of the rate of change of angular momentum equation (2.14) is

$$\frac{\partial\hat{Q}}{\partial\eta} = O(\epsilon^3), \tag{4.9}$$

so that $\hat{Q}$ remains constant through the inner layer at least to $O(\epsilon^2)$. Further, since the torque must be continuous from the outer to the inner solution we deduce from (2.21), and the fact that $(\mathbf{b}\cdot\mathbf{n}') \sim O(\epsilon^3)$ in the outer solution, that

$$\hat{Q} = \epsilon\frac{\kappa}{\mathcal{V}} + O(\epsilon^5). \tag{4.10}$$

From (2.16) we deduce that

$$\mathbf{r}_{0\eta}\cdot\mathbf{r}_{0\eta} = 1, \quad \hat{T}_1 = \gamma(\mathbf{r}_{0\eta}\cdot\mathbf{r}_{1\eta}), \tag{4.11}$$

and

$$\hat{\mathbf{t}} = \mathbf{r}_{0\eta} + \epsilon[\mathbf{r}_{1\eta} - (\mathbf{r}_{1\eta}\cdot\mathbf{r}_{0\eta})\mathbf{r}_{0\eta}] + \cdots,$$

$$\hat{\mathbf{t}}_0\cdot\hat{\mathbf{V}}_0 = \mathbf{r}_{0\eta}\cdot\hat{\mathbf{V}}_0 = 0, \quad \mathbf{r}_{0\eta}\cdot\hat{\mathbf{V}}_1 = -\hat{\mathbf{t}}_1\cdot\hat{\mathbf{V}}_0, \tag{4.12}$$

where $(\,)_\eta = \partial(\,)/\partial\eta$.

4.2.1 The leading-order inner equations

The leading-order inner equation for the rate of change of linear momentum obtained from (2.13) is

$$\left(\hat{T}_0\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_0\right)_\eta = \mathbf{0}, \tag{4.13}$$

and forming the vector product between $\mathbf{r}_{0\eta}$ and the leading-order term from the rate of change of angular momentum Eq. 2.22 gives us

$$\hat{\mathbf{V}}_0 = \frac{\kappa}{\mathcal{V}}(\mathbf{r}_{0\eta} \times \mathbf{r}_{0\eta\eta}) + \mathbf{r}_{0\eta} \times (\mathbf{r}_{0\eta} \times \mathbf{r}_{0\eta\eta\eta}). \tag{4.14}$$

The boundary conditions for the solution of the inner equations are determined by the matching conditions given in Appendix A as follows.

First, on substitution of the outer solutions in the matching condition (6.3), we obtain for the $O(\epsilon)$ matching between regions 1 and 2:

$$\lim_{\substack{\epsilon\to 0\\ \xi\,\text{fixed}}} \left\{\left[\left(U\sin\lambda t + \bar{U}\cos\lambda t\right)\mathbf{i} + \left(U\cos\lambda t - \bar{U}\sin\lambda t\right)\mathbf{j}\right]\sin\pi s_b + \mathbf{r}_0\left(\frac{\delta\xi}{\epsilon}\right)\right.$$
$$\left. - \left[\left(U_1\sin\lambda_1 t + \bar{U}_1\cos\lambda_1 t\right)\mathbf{i} + \left(U_1\cos\lambda_1 t - \bar{U}_1\sin\lambda_1 t\right)\mathbf{j}\right]\sin\alpha s_b - \left(\frac{\delta\xi}{\epsilon} + a s_b\right)\mathbf{k}\right\} = \mathbf{0},$$

and for the $O(\epsilon)$ matching between region 2 and region 3:

$$\lim_{\substack{\epsilon\to 0\\ \xi\,\text{fixed}}} \left\{\left[\left(U\sin\lambda t + \bar{U}\cos\lambda t\right)\mathbf{i} + \left(U\cos\lambda t - \bar{U}\sin\lambda t\right)\mathbf{j}\right]\sin\pi s_b + \mathbf{r}_0\left(\frac{\delta\xi}{\epsilon}\right)\right.$$
$$\left. - \left[\left(U_3\sin\lambda_3 t + \bar{U}_3\cos\lambda_3 t\right)\mathbf{i} + \left(U_3\cos\lambda_3 t - \bar{U}_3\sin\lambda_3 t\right)\mathbf{j}\right]\sin\beta(1-s_b) - \left[\frac{\delta\xi}{\epsilon} + b(1-s_b)\right]\mathbf{k}\right\} = \mathbf{0}.$$

In order to accomplish the matching of the $\mathbf{i}$ and $\mathbf{j}$ components we set $U_1 = U_3 = U$, $\bar{U}_1 = \bar{U}_3 = \bar{U}$, $\alpha = \beta = \pi$ and $\lambda_1 = \lambda_3 = \lambda$. The matching of the $\mathbf{k}$ components in the above expression can be expressed in a simpler form by setting $\delta\xi/\epsilon = \eta$:

$$\lim_{\eta\to-\infty}[\mathbf{r}_0(\eta) - \eta\mathbf{k}] = a s_b\mathbf{k}, \quad \lim_{\eta\to+\infty}[\mathbf{r}_0(\eta) - \eta\mathbf{k}] = b(1-s_b)\mathbf{k}. \tag{4.15}$$

Similarly, for the other variables we deduce the leading-order matching conditions

$$\lim_{\eta\to\pm\infty}\mathbf{r}_{0\eta}(\eta) = \mathbf{k}, \quad \lim_{\eta\to\pm\infty}\hat{T}_0(\eta) = T_0,$$
$$\lim_{\eta\to\pm\infty}[\mathbf{r}_{0\eta}(\eta) \times \mathbf{r}_{0\eta\eta}(\eta)] = \mathbf{0}, \quad \lim_{\eta\to\pm\infty}\hat{\mathbf{V}}_0(\eta) = \mathbf{V}_0(s_b) = \mathbf{0}, \tag{4.16}$$

where the value of $\mathbf{V}_0(s_b)$ is determined by (3.5). With these boundary conditions Eq. 4.13 can be integrated to give

$$\hat{T}_0\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_0 = T_0\mathbf{k}. \tag{4.17}$$

When $\hat{\mathbf{V}}_0$ is eliminated between this equation and (4.14), and the vector product of the resulting equation and $\mathbf{r}_{0\eta}$ is formed, the result is

$$(\mathbf{r}_{0\eta} \times \mathbf{r}_{0\eta\eta})_\eta + \frac{\kappa}{\mathcal{V}}\mathbf{r}_{0\eta\eta} = -T_0\mathbf{r}_{0\eta} \times \mathbf{k}. \tag{4.18}$$

Equation (4.18) can now be integrated subject to the above boundary conditions and the constant of integration is found to be $(\kappa/\mathcal{V})\mathbf{k}$. When a further cross-product of this final equation with $\mathbf{r}_{0\eta}$ is formed, the equation

$$\mathbf{r}_{0\eta\eta} + \frac{\kappa}{\mathcal{V}}(\mathbf{r}_{0\eta} \times \mathbf{k}) - T_0\left[(\mathbf{r}_{0\eta}\cdot\mathbf{k})\mathbf{r}_0 - (\mathbf{r}_0\cdot\mathbf{r}_{0\eta})\mathbf{k}\right] = \mathbf{0} \tag{4.19}$$

is obtained.

We now introduce a local cylindrical coordinate system $(\hat{r}, \hat{\theta}, \hat{z})$ with its origin at $\mathbf{R}(s_b)$ and unit basis vectors $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \mathbf{k})$ where

$$\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta = \mathbf{k}.$$

Thus, setting $\mathbf{r}_0 = \hat{r}\hat{\mathbf{e}}_r + \hat{z}\mathbf{k}$ we find that the components of this equation are

$$\hat{r}_{\eta\eta} - \hat{r}(\hat{\theta}_\eta)^2 - T_0\hat{z}_\eta\hat{r} = -\frac{\kappa}{\mathcal{V}}\hat{r}\hat{\theta}_\eta, \tag{4.20}$$

$$(2\hat{r}_\eta\hat{\theta}_\eta + \hat{r}\hat{\theta}_{\eta\eta}) = \frac{\kappa}{\mathcal{V}}\hat{r}_\eta, \tag{4.21}$$

$$\hat{z}_{\eta\eta} + \frac{1}{2}T_0(\hat{r}^2)_\eta = 0. \tag{4.22}$$

Now $\hat{r}$ times (4.21) can be integrated, subject to the above boundary conditions at $\eta \to \pm\infty$, to give $\hat{\theta}_\eta = \kappa/(2\mathcal{V})$, and (4.22) can be integrated to give

$$\hat{z}_\eta = 1 - \frac{1}{2}T_0\hat{r}^2. \tag{4.23}$$

When these results are used to eliminate $\hat{\theta}_\eta$ and $\hat{z}_\eta$ from (4.20) we obtain a Duffing equation for $\hat{r}$:

$$\hat{r}_{\eta\eta} - \left[T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2\right]\hat{r} + \frac{1}{2}T_0^2\hat{r}^3 = 0. \tag{4.24}$$

The solution of this equation subject to the boundary condition $\lim_{\eta\to\pm\infty}\hat{r} = 0$ is

$$\hat{r} = \frac{1}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\ \mathrm{sech}\left\{\left(\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\right)\eta\right\}, \tag{4.25}$$

provided the condition

$$\frac{\kappa}{\mathcal{V}\sqrt{T_0}} < 2 \tag{4.26}$$

is satisfied, otherwise $\hat{r}(\eta) \equiv 0$. If the inequality in (4.26) is replaced by an equality, this gives the critical condition for the localised buckling to occur in the whirling yarn.

When this final result is inserted in (4.23) and this equation is integrated subject to the condition that $\hat{z} = 0$ when $\eta = 0$, the result is

$$\hat{z} = \eta - \frac{1}{2T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\ \tanh\left\{\left(\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\right)\eta\right\}. \tag{4.27}$$

If we now apply the matching conditions (4.15) to the $\mathbf{k}$ component of $\mathbf{r}_0$ we find that

$$\frac{1}{2T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2} = as_b, \quad \frac{1}{2T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2} = b(s_b - 1). \tag{4.28}$$

Thus

$$as_b = -b(1 - s_b). \tag{4.29}$$

Two further equations for the determination of $a, b, \rho_1, \rho_3, s_b$ can be obtained if we consider the $O(\epsilon^2)$ inner equation of linear momentum and make further use of the matching conditions.

4.2.2 The $O(\epsilon^2)$ equations

From the equation of linear momentum we obtain

$$\left[\hat{T}_0\hat{\mathbf{t}}_1 + \hat{T}_1\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_1\right]_\eta = \mathbf{0}. \tag{4.30}$$

The solution of this equation is subject to the constraints (4.11) and (4.12). Equation (4.30) can be integrated and the constants of integration can be determined from the matching conditions. (Note that results (3.5) still hold for this outer solution.) Thus, matching between regions 1 and 2, using (4.16), we obtain

$$\hat{T}_0\hat{\mathbf{t}}_1 + \hat{T}_1\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_1 = T_0\left[(U\sin\lambda t + \bar{U}\cos\lambda t)\mathbf{i} + (U\cos\lambda t - \bar{U}\sin\lambda t)\mathbf{j})\right]\pi\cos\pi s_b + \gamma a\mathbf{k}, \tag{4.31}$$

and matching between regions 2 and 3 we obtain

$$\hat{T}_0\hat{\mathbf{t}}_1 + \hat{T}_1\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_1 = T_0\left[(U\sin\lambda t + \bar{U}\cos\lambda t)\mathbf{i} + (U\cos\lambda t - \bar{U}\sin\lambda t)\mathbf{j}\right]\pi\cos\pi s_b - \gamma b\mathbf{k}. \tag{4.32}$$

Since the right-hand sides of these last two equations must be identical, we set $a = -b$, and inserting this result into (4.29) and using results (4.28), we obtain

$$s_b = \frac{1}{2}, \quad a = -b = \frac{1}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}, \tag{4.33}$$

and finally the first integral of (4.30) is

$$\hat{T}_0\hat{\mathbf{t}}_1 + \hat{T}_1\mathbf{r}_{0\eta} + \hat{\mathbf{V}}_1 = \left\{\frac{\gamma}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\right\}\mathbf{k}. \tag{4.34}$$

Finally we summarise the $O(\epsilon)$ outer solution.

4.2.3 The $O(\epsilon)$ outer solution

$$\begin{aligned}
u_1 &= \left[U\sin\lambda t + \bar{U}\cos\lambda t\right]\sin\pi s, \quad v_1 = \left[U\cos\lambda t - \bar{U}\sin\lambda t\right]\sin\pi s,\\
w_1 &= \left[\frac{1}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\right]s, \quad 0 \le s < \frac{1}{2}\\
&= \left[\frac{1}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}\right](s-1) \quad \frac{1}{2} < s \le 1,\\
T_1 &= \frac{\gamma}{T_0}\sqrt{T_0 - \left(\frac{\kappa}{2\mathcal{V}}\right)^2}.
\end{aligned} \tag{4.35}$$

where

$$T_0\pi^2 = (\lambda - 1)^2.$$

We note that the expressions for u_1, v_1 and T_0 are identical to the *case 1* ballooning results (3.12) but that now w_1 and T_1 are no longer zero because of the snarling. Note also that the snarl forms in the centre of the balloon at $s = 1/2$.

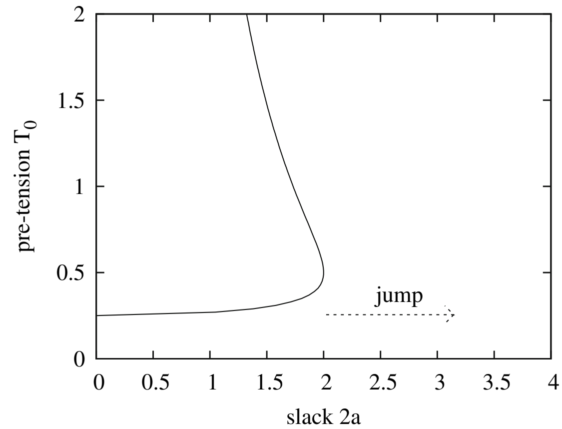

Fig. 3 The snarling bifurcation diagram (taking $\kappa/\mathcal{V} = 1$)

4.3 Discussion

To illustrate the snarling bifurcation, in Fig. 3 we plot T_0 against the static strain $2a$, which represents the slack taken up by the localised snarl. Note that the bifurcation is subcritical. Snarling is initiated when the pre-tension T_0 drops to the critical value given by (4.26). As the instability is highly localised, located away from the ends of the rod, snarling effectively takes place under dead-loading conditions. This means that, rather than following the post-buckling curve in Fig. 3, a dynamical jump will occur to a remote (self-contacting) state, as indicated by the arrow in the figure.

Comparing the critical load for ballooning, (3.16), with the critical load for snarling, (4.26), we conclude that snarling may be initiated before or after ballooning, depending on whether $\mathcal{V} < \pi$ or $\mathcal{V} > \pi$. In the former case snarling occurs off the straight yarn path.

In interpreting these critical loads it should be remembered that the limit $\mathcal{V} \to 0$ (or $\tilde{V} \to 0$) is singular: with no yarn feeding through, the spindle acts to impart an unlimited amount of twist to the yarn.

5 Concluding remarks

In this paper, we have derived a theory for the snarling instability that can occur in textile yarn manufacturing systems. Within this theory the well-known Coyne [10] solution for the localised buckling of a twisted isotropic elastic rod appears as an internal bending-layer solution. The particular system we have chosen to illustrate the theory is an idealised version of a false-twist system. In such a system the instability will occur in the uptwist region where the torque is high and we have confined our analysis to this region as the torque is negligible in the post twisting-spindle region. We have simplified the boundary conditions at the spindle and our calculation of the torque/twist relation (2.21) is a result of this idealisation.

One of the main systems where such instabilities can cause problems is the false-twist yarn-texturing system [1]. In these systems the yarn mechanics in the uptwist region is complicated by the heating elements that are used to heat-set the twist in the thermo-plastic fibres used in such manufactures. As the yarn passes over these elements, its elastic properties will be changed so that our theory, which ignores these effects, can only approximate the instabilities that occur in such systems. However, we believe that snarling instabilities are essentially mechanical in nature and that changes in the physical properties of the fibres as they pass over the heating elements may cause a critical combination of parameters to trigger a snarling instability.

6 Appendix A: Formal discussion of the matching procedure

The matching procedure provides boundary conditions for the solutions in the various sections. As all variables must be continuous at all points on the yarn axis, we shall illustrate the matching procedure for the position vectors between regions 1 and 2. Let the region 1 outer solution have the form

$$\mathbf{R}(s) = \mathbf{k}s + \epsilon \mathbf{R}_1(s) + \cdots,$$

and let the inner solution in region 2 have the form

$$\mathbf{R}(s) = \mathbf{R}(s_b) + \epsilon \mathbf{r}_0(\eta) + \epsilon^2 \mathbf{r}_1(\eta) + \cdots = \mathbf{k}s_b + \epsilon \mathbf{R}_1(s_b) + \epsilon \mathbf{r}_0(\eta) + \epsilon^2 \mathbf{r}_1(\eta) + \cdots,$$

where $\eta = (s - s_b)/\epsilon$ is the stretched inner layer independent variable, and the inner layer is centred at $\mathbf{R}(s_b)$, given in this case by (4.1).

Now introduce the intermediate matching variable:

$$\xi = \frac{s - s_b}{\delta(\epsilon)} = \frac{\epsilon}{\delta(\epsilon)}\eta, \tag{6.1}$$

where the order of magnitude of the small parameter $\delta(\epsilon) > 0$ is such that

$$\lim_{\epsilon \to 0} \delta(\epsilon) = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \frac{\epsilon}{\delta} = \lim_{\epsilon \to 0} \frac{\delta^2}{\epsilon} = 0. \tag{6.2}$$

That is to say that δ tends to zero slower than ϵ but faster than $\sqrt{\epsilon}$. Now rewrite the inner solution in terms of $\eta = \delta\xi/\epsilon$, and expand the region 1 outer solution in the neighbourhood of $s = s_b$ using the substitution $s = s_b + \delta\xi$.

We match the region 1 solution to the inner solution using the following limit procedure:

$$\lim_{\substack{\epsilon \to 0 \\ \xi \text{ fixed}}} \left\{ \mathbf{R}(s_b) + \epsilon \mathbf{r}_0\left(\frac{\delta\xi}{\epsilon}\right) + \epsilon^2 \mathbf{r}_1\left(\frac{\delta\xi}{\epsilon}\right) + \cdots \right.$$
$$\left. - \mathbf{R}_0(s_b + \delta\xi) - \epsilon \mathbf{R}_1(s_b + \delta\xi) + \cdots \right\} = \mathbf{0}, \tag{6.3}$$

where Taylor-series expansions of the outer solution terms about $s = s_b$ are to be used before the limit process is applied. A similar expression applies between the inner solution and the region 3 outer solution, and the other variables can be matched in similar manner.

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